On the Construction of a Jacobi Matrix from Mixed Given Data

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

Submitted by Hans Schneider

ABSTRACT

It is shown that a $n \times n$ Jacobi matrix is uniquely determined by its n eigenvalues and by the selected set of $n-1$ entries in the matrix.

Let J denote a Jacobi matrix—that is, a tridiagonal matrix of the form

$$J = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{bmatrix} \tag{1}$$

where all a_i, b_i are real and all b_i positive. Let J_τ denote the principal submatrix of J obtained by deleting either the first row and first column of J or alternatively the last row and last column of J. It is well known [1] that a knowledge of the spectra of J and J_τ uniquely determines J. If J is persymmetric, however, a knowledge of only the spectrum of J uniquely determines J.

The purpose of this note is to prove the following theorem.

THEOREM. *Let J be a Jacobi matrix as in* (1), *with all a_i, b_i real and all b_i positive. Suppose we are given its n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,*

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as well as the $n-1$ *entries* $a_1, a_2, \ldots, a_{[n/2]}, b_1, b_2 \ldots, b_{[n-1/2]}$, *(where* $[x]$ *denotes the greatest integer less than or equal to* x*). Then these data determine a unique Jacobi matrix.*

N.B. The theorem does not assert the existence of such a matrix, but the uniqueness, if it exists.

Proof. The proof will be modeled on that given in [1]. We let

$$B(\lambda) = \prod_1^n (\lambda - \lambda_i), \qquad A(\lambda) = \prod_1^{n-1} (\lambda - \mu_i), \tag{2}$$

where the μ_i are the eigenvalues of J_r, obtained by deleting the first row and first column of J. As was shown in [1],

$$\frac{A(\lambda)}{B(\lambda)} = \left((\lambda - J)^{-1}\delta_0, \delta_0\right), \tag{3}$$

where $\delta_0 = (1,0,0,\ldots,0)^T$. This is a simple consequence of Cramer's rule. By decomposing the left side of (3) into partial fractions and then into a geometric series in $1/\lambda$ we find

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^n \frac{\alpha_k}{\lambda - \lambda_k} = \sum_{k=1}^n \sum_{j=0}^\infty \frac{\lambda_k^j \alpha_k}{\lambda^{j+1}} = \sum_{j=0}^\infty \frac{(J^j\delta_0, \delta_0)}{\lambda^{j+1}}. \tag{4}$$

By comparing terms in (4) we find

$$\begin{aligned} \sum_{k=1}^n \alpha_k &= 1, \\ \sum_{k=1}^n \lambda_k \alpha_k &= (J\delta_0, \delta_0) = a_1, \\ \sum_{k=1}^n \lambda_k^2 \alpha_k &= \left(J^2\delta_0, \delta_0\right) = a_1^2 + b_1^2, \\ &\vdots \\ \sum_{k=1}^n \lambda_k^j \alpha_k &= (J^j\delta_0, \delta_0), \\ &\vdots \\ \sum_{k=1}^n \lambda_k^{n-1} \alpha_k &= \left(J^{n-1}\delta_0, \delta_0\right). \end{aligned} \tag{5}$$

The above linear system of n equations in the n unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$ must have a solution, since its determinant is the Vandermonde determinant, which does not vanish by virtue of the distinctness of the λ_k. The right sides are also known by the hypotheses of the theorem. Clearly $(J^j\delta_0, \delta_0)$ depends only on $a_1, a_2, \dots, a_{[(j+1)/2]}, b_1, b_2, \dots, b_{[j/2]}$, so that the α_k are uniquely determined from given data.

Now we can return to (5) and determine $A(\lambda)$ or equivalently the μ_k. By the result of [1] a knowledge of the λ_k and μ_k determines a unique J. ∎

A comparable result can be proved for inverse Sturm-Liouville operators. There the potential $q(x)$ is given over half the interval, and one spectrum is given. Together these determine a unique $q(x)$ on the full interval. For details we refer to [2].

REFERENCES

1 H. Hochstadt, On the construction of a Jacobi matrix from spectral data, *Linear Algebra and Appl.* 8:435–446 (1974).
2 H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, *SIAM J. Appl. Math.* 34:676–680 (1978).

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